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V-SUPER AND *E*-SUPER VERTEX-MAGIC TOTAL LABELING OF GRAPHS

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ABSTRACT

Let G be a graph of order p and size q. A vertex-magic total labeling is an assignment of the integers 1, 2, . . . , p + q to the vertices and the edges of G, so that at each vertex, the vertex label and the labels on the edges incident at that vertex, add to a fixed constant, called the magic constant of G. Such a labeling is V-super vertex-magic total if f(V(G) = {1, 2, . . . , p}, and is an E-super vertex-magic total if f(E(G) = {1, 2, . . . , q}. A graph that admits a V-super vertex-magic total labeling is called V-super vertex-magic total. Similarly, a graph that admits an E-super vertex-magic total labeling is called E-super vertex-magic total. In this paper, we provide some properties of E-super vertex-magic total labeling of graphs and we prove V-super and E-super vertex-magic total labeling of the product of cycles $C_m \times C_n$*, where* $m, n \geq 3$ *and m, n odd.*

KEYWORDS: *Vertex Magic Total Labeling, V-Super Vertex Magic Total Labeling, E- Super Vertex Magic Total Labeling*

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1. INTRODUCTION

In this paper, we consider only finite simple undirected graph. The set of vertices and edges of a graph *G* will be denoted by *V(G)* and *E(G)* respectively and we let $p = |V(G)|$ and $q = |E(G)|$. The set of neighbors of a vertex *v* is denoted by *N(v).* For general graph theoretic notations, we follow [18].

A *labeling* of a graph *G* is a mapping that carries a set of graph elements, usually the vertices and/or edges, into a set of numbers, usually, integers, called *labels*. Many kinds of labeling have been studied and an excellent survey of graph labeling can be found in [4].

MacDougal et al. [13] introduced the notion of vertex-magic total labeling. For a graph *G* with *p* vertices and *q* edges, a vertex-magic total labeling (VMTL) is a bijection $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots , p + q\}$ such that for every vertex $u \in V(G)$, its weight $wt_f(u) = f(u) + \sum_{v \in N(u)} f(uv) = k$ for some constant *k* . This constant is called the magic constant of the VMTL. They studied the basic properties of vertex-magic graphs and showed some families of graphs having a vertex -magic total labeling

MacDougall et al [14] and Swaminathan and Jeyanthi [21] introduced different labeling with same name super vertex-magic total labeling. To avoid confusion, Marimuthu and Balakrishnan [15] called a vertex-magic total labeling is

***E*-super vertex magic total** if $f(E(G)) = \{1, 2, 3, \ldots, q\}$, i.e. the smallest labels are assigned to the edges. A graph G is called ***E*-super vertex-magic total** if it admits an *E*-super vertex-magic total labeling.

MacDougall, Miller and Sugeng [14] introduced the notion of super vertex-magic total labeling, A vertex-magic total labeling is super if $f(V(G)) = \{1, 2, 3, \ldots, p\}$, we call it as ***V*-super vertex-magic total labeling**. A graph G is called ***V*-super vertex-magic total** if it admits a *V*-super vertex-magic total labeling, i.e. the smallest labels are assigned to the vertices. In [14], they proved that an *r*-regular graph of order *p* has a *V*-super vertex-magic total labeling then *p* and *r* have opposite parity and if (i) $p \equiv 0 (mod\ 8)$ then $q \equiv 0 (mod\ 4)$, (ii) $p \equiv 4 (mod\ 8)$ then $q \equiv 2 (mod\ 4)$. The cycle C_n has a *V*-super vertex-magic total labeling if and only if *n* is odd. They also conjectured that if $n \equiv 0(mod\ 4)$; $n > 4$, then K_n has a *V*-super vertex-magic total labeling. But this conjecture was proved by J. Gomez in [5] also a tree, wheel, fan, ladder, or friendship graph has no *V*-super vertex-magic total labeling. If *G* has a vertex of degree one, then *G* is not *V*-super vertex-magic total. For more results regarding *V*-super VMTLs, see [4], [5] and [18].

Swaminathan and Jeyanthi [21] showed that a path P_n is *E*-super vertex-magic total if and only if *n* is odd and $n \geq 3$. A cycle C_n is *E*-super vertex-magic if and only if *n* is odd. mC_n is *E*-super vertex-magic total if and only if *m* and *n* are odd. Marimuthu and Balakrishnan [15] proved that for a connected graph *G* and *G* has an *E*-super vertex-magic total labeling with magic constant *k* then $k \geq (5p-3)/2$. Also, proved for a (p, q) graph, with even *p* and $q = p - 1$ or *p*, then the graph is not *E*-super vertex-magic total. Generalized Petersen graph $P(n, m)$ is not *E*-super vertex-magic total if *n* is odd. They also discussed about the *E*-super vertex magicness of *m* connected graph $H_{m,n}$. A graph with the odd order can be decomposed into two Hamiltonian cycles, then *G* is *E*-super vertex-magic total. A graph *G* can be decomposed into two spanning subgraphs G_1 and G_2 where G_1 is *E*-super vertex-magic total and G_2 is magic and regular then *G* is *E*-super vertex-magic total. Also, they proved as the two spanning subgraphs are *E*-super vertex-magic total and one is regular then the graph *G* is *E*-super vertex-magic total. Readers are referred to [6, 7, 8, 9, 10, 12, 20, and 22] for general background and basic constructions regarding *E*-super VMTLs.

The following results will be very useful to prove some theorems.

Lemma1.2[14]

If a non- trivial graph *G* is an *V*-super vertex magic total, then the magic constant *h* is given by $h = 2q + \frac{p+1}{2} + \frac{q(q+1)}{p}$.

Lemma1.3[21]

If a non- trivial graph *G* is an *E*-super vertex magic total, then the magic constant *k* is given by $k = q + \frac{p+1}{2} + \frac{q(q+1)}{p}$.

Theorem 1.4[17]

The dual of an *E*-super (respectively *V*-super) vertex-magic total labeling for a graph *G* is a *V*-super (respectively *E*-super) vertex-magic total labeling if and only if *G* is *r*- regular, $r \geq 1$.

Theorem 1.5[14]

No complete bipartite graph is *V*-super vertex magic total.

Theorem 1.6[17]

Let G be any graph without isolated vertex. If G is E-super vertex-magic total, then the magic constant $k > p + q$.

Theorem 1.7[20]

No E-super vertex-magic total graph has two or more isolated vertices or an isolated edge.

Theorem 1.8[1]

Complete bipartite graph $K_{m,n}$ is Hamiltonian if and only if $m = n$.

Theorem 1.9[16]

Let G be an E-super vertex-magic total graph with p vertices, q edges, and magic constant k. Then the degree d of any vertex of G satisfies

$$q + \frac{1}{2} - \sqrt{(q + \frac{1}{2})^2 - 2(k - p - q)} \leq d \leq \frac{-1}{2} + \sqrt{2(k - p) - \frac{7}{4}}.$$

This article contains four sections. In section 1, a brief history of the subject is given. Section 2 establishes some properties of E-super vertex-magic total labeling of graphs. In section 3, we provide V-super and E-super vertex-magic total labeling of the product of cycles $C_m \times C_n$, where $m, n \geq 3$ and m, n odd.

2. SOME PROPERTIES OF *E*-SUPER VERTEX MAGIC TOTAL LABELING OF GRAPHS

In this section, we give some properties of E-super vertex-magic total labeling of graphs.

Theorem 2.1

Let G be an E-super vertex-magic total graph with one isolated vertex. Then the order p and the size q must satisfy $(p - 1)^2 + p^2 = (2q + 1)^2$.

Proof

Since G is an E-super vertex-magic total graph, G cannot have more than one isolated vertex. Suppose that G has an isolated vertex, say u. Since the label for any vertex is at most $p + q$, then the weight of u satisfies $wt_f(u) = k \leq (p + q)$. If $wt_f(u) < (p + q)$, then we can find another vertex v, different from u, with label $p + q$, so that $k = wt_f(v) > (p + q)$. This is a contradiction to the assumption that $k < (p + q)$. Hence $wt_f(u) = k = (p + q)$.

But from the Lemma 1.3, $k = q + \frac{p+1}{2} + \frac{q(q+1)}{p}$

$$p + q = q + \frac{p+1}{2} + \frac{q(q+1)}{p}$$

$$p = \frac{p+1}{2} + \frac{q(q+1)}{p}$$

$$\frac{p}{2} = \frac{1}{2} + \frac{q^2}{p} + \frac{q}{p}.$$

$$p = 1 + \frac{2q^2}{p} + \frac{2q}{p}$$

$$p(p - 1) = 2q(q + 1)$$

Multiplying by 2 on both sides and simplifying we get, $(p - 1)^2 + p^2 = (2q + 1)^2$. ■

Next, we consider the complete bipartite graphs. It was shown in [13] that the only complete bipartite graphs that could admit a VMTL are $K_{m,m}$ and $K_{m,m+1}$ and that a VMTL does exist in both cases for every $m > 1$. So these are the only candidates remain to discuss the E- super VMTL. However, we show that neither of them is E-super vertex-magic total in the following theorem.

Theorem 2.2.

No complete bipartite graph except $K_{1,2}$ is an E-super vertex-magic total graph.

Proof

Obviously, $K_{1,1}$ is isomorphic to P_2,which is not E-super vertex magic total by Theorem 1.7. Suppose, by way of contradiction, that $K_{m,m}$ has an E-super vertex-magic total labeling. Since $K_{m,m}$ is regular,we get $K_{m,m}$ is V-super vertex -magic total by the method of duality (see Theorem 1.4). Note that, in view of Theorem 1.5, no complete bipartite graph is V-super vertex-magic total, a contradiction.

Now we suppose that $K_{m,m+1}$ is E-super vertex-magic total with $p = 2m + 1$, $q = m(m + 1)$. Then according to the Lemma 1.3, the magic constant k is given by

$$k = q + \frac{p+1}{2} + \frac{q(q+1)}{p} = m\,(m + 1) + \frac{2m+2}{2} + \frac{m(m+1)[m(m+1)+1]}{2m+1}$$

It follows that

$$k = \frac{m^4+4m^3+7m^2+5m+1}{2m+1}.$$

Now we consider the following cases according to the nature of m.

Case (i) If m is odd, let $m = 2r - 1$, then by Lemma 1.3, the magic constant k is given by

$$k = \frac{(2r-1)^4+4(2r-1)^3+7(2r-1)^2+5(2r-1)+1}{2(2r-1)+1} = \frac{16r^4+4r^2-2r}{4r-1} = 4r^3 + r^2 + r + \frac{r(r-1)}{4r-1}$$ which is an integer only when $r = 1$.

If $r = 1$, then we get $K_{1,2}$. Clearly $K_{1,2}$ is an E-super vertex-magic total graph with magic constant $k = 6$. It is shown in Figure 1.

Case (ii) If m is even, let $m = 2r$, then by Lemma 1.3, the magic constant k is given by

$$k = \frac{(2r)^4+4(2r)^3+7(2r)^2+5(2r)+1}{2(2r)+1} = \frac{16r^4+32r^3+28r^2+10r+1}{4r+1} = 4r^3 + 7r^2 + 6r + 1 - \frac{3r^2}{4r+1}$$ which is not an integer.

In all the cases, we get k is not an integer, and this concludes the proof. ■

Corollary 2.3.

No complete bipartite Hamiltonian graph is E-super vertex-magic total.

Proof.

Let $G \cong K_{m,n}$ be a complete bipartite Hamiltonian graph. If $m = 1$, then by Theorem 1.6, n should be equal to 1. But $K_{1,1}$ is isomorphic to P_2,which is not E-super vertex-magic total by Theorem 1.7. If $m > 1$, then by above theorem, G is not an E-super vertex-magic total graph. ■

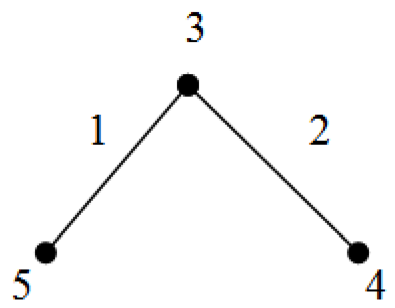

Figure 1: *E*-super Vertex Magic Total Labeling of $K_{1,2}$ with $k = 6$

Theorem 2.4.

If G is a connected planar graph with $p \geq 3$ and G has an E-super vertex magic total labeling with magic constant k, then $k \leq \frac{25p^2-77p+60}{2p}$.

Proof.

If G is a connected planar graph with $p \geq 3$, then $q \leq 3p - 6$

By Lemma 1.3, the magic constant $k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$

$$\leq 3p - 6 + \frac{p+1}{2} + \frac{(3p-6)(3p-5)}{p}$$

$$= \frac{7p-11}{2} + \frac{9p^2-33p+30}{p}$$

$$= \frac{25p}{2} - \frac{77}{2} + \frac{30}{p}$$

$$k \leq \frac{25p^2-77p+60}{2p}. \blacksquare$$

Theorem 2.5.

If G is a connected maximal planar graph with $p \geq 3$ and G has an E-super vertex magic total labeling with magic constant k, then $k = \frac{25p^2-77p+60}{2p}$.

Proof.

If G is a connected maximal planar graph with $p \geq 3$, then $q = 3p - 6$

By Lemma 1.3, the magic constant $k = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$

$$= 3p-6 + \frac{p+1}{2} + \frac{(3p-6)(3p-5)}{p}$$

$$= \frac{7p-11}{2} + \frac{9p^2-33p+30}{p}$$

$$= \frac{25p}{2} - \frac{77}{2} + \frac{30}{p}$$

$$k = \frac{25p^2-77p+60}{2p}. \blacksquare$$

3. *V*-SUPER AND *E*-SUPER VERTEX MAGIC TOTAL LABELING OF PRODUCT OF CYCLES

For C_m x C_n, $p=mn$ and $q=2mn$. In this section we provide *V*-super and *E*-super vertex magic total labeling of product of cycles C_m x C_n , where $m, n \geq 3$ and m, n odd.

Theorem 3.1

For each $m,n \geq 3$ and m,n odd, there exists a *V*-super vertex magic labeling of $C_m x C_n$ with the magic constant $k = \frac{17}{2}mn + \frac{5}{2}$.

Proof

Let $G \cong C_m$ x C_n have vertices $v_{i,j}$, vertical edges $v_{i,j}v_{i+1,j}$ and horizontal edges $v_{i,j}v_{i,j+1}$ where $i=0,1,\ldots,m-1$, $j=0,1,\ldots,n-1$ and m and n are odd integers greater than 1. Consider the following labeling, where the subscripts i and j are taken modulo m and n, respectively.

$$f(v_{i,j}) = jm+1+i$$

$$f(v_{i,j}\, v_{i+1,j}) = \begin{cases} (2n-j-1)m+1+\frac{i}{2} \text{ if } i \text{ is even} \\ (2n-j-1)m+1+\frac{i+m}{2} \text{ if } i \text{ is odd} \end{cases}$$

$$f(v_{i,j}\, v_{i,j+1}) = \begin{cases} \left(2n+\frac{j}{2}+1\right)m-i \text{ if } j \text{ is even} \\ \left(2n+\frac{j+n}{2}+1\right)m-i \text{ if } j \text{ is odd} \end{cases}$$

The magic constant h is obtained from the following cases.

Case(i) If both i and j are even then the magic constant is given by

$$h= f(v_{i,j})+f(v_{i-1,j}\, v_{i,j})+f(v_{i,j}\, v_{i+1,j})+f(v_{i,j-1}\, v_{i,j})+f(v_{i,j}\, v_{i,j+1})$$

$$=[jm+1+i]+[(2n-j-1)m+1+\frac{m+i-1}{2}]+[(2n-j-1)m+1+\frac{i}{2}]+[(2n+\frac{n+j-1}{2}+1)m- i]+[(2n+\frac{j}{2}+1)m-i]$$

$$h = \frac{17}{2}mn+\frac{5}{2}$$

Case(ii) If i is even and j is odd then the magic constant h is given by

$$h = f(v_{i,j})+f(v_{i-1,j}\, v_{i,j})+f(v_{i,j}\, v_{i+1,j})+f(v_{i,j-1}\, v_{i,j})+f(v_{i,j}\, v_{i,j+1})$$

$$=[jm+1+i]+[(2n-j-1)m+1+\frac{m+i-1}{2}]+[(2n-j-1)m+1+\frac{i}{2}]+[(2n+\frac{j-1}{2}+1)m-i]+[(2n+\frac{n+j}{2}+1)m-i]$$

$$h = \frac{17}{2}mn+\frac{5}{2}.$$

Similarly, we can prove for i is odd, j is even and for both i and j are odd.

In all the cases, the magic constants are same and $f(V(G))=\{1,2,3,\ldots,p\}$ and $f(E(G))= \{p+1,p+2,\ldots,p + q\}$.Hence G is *V*-super vertex magic total . ■

Corollary 3.2

For each $m, n \geq 3$ and m, n odd, there exists an *E*-super vertex magic total labeling of C_m x C_n with the magic constant $k = \frac{15}{2}mn + \frac{5}{2}$.

Proof

By the above Theorem, $G \cong C_m \times C_n$ is V-super vertex-magic total for $m, n \geq 3$ and m, n odd. Since G is regular, by duality method, G is E-super vertex magic total for $m, n \geq 3$ and m, n odd. ■

Corollary 3.3

For each $m, n \geq 3$ and m even, there does not existan E-super vertex-magic total labeling of $C_m \times C_n$.

Corollary 3.4

For each $m, n \geq 3$ and m odd, n even, there does not exist an E-super vertex-magic total labeling of $C_m \times C_n$.

Corollary 3.5

For each $m, n \geq 3$ and m even, there does not exist a V-super vertex-magic total labeling of $C_m \times C_n$.

Corollary 3.6

For each $m, n \geq 3$ and m odd, n even, there does not exists a V-super vertex magic total labeling of $C_m \times C_n$.

An example is given in the following Figure 3.1

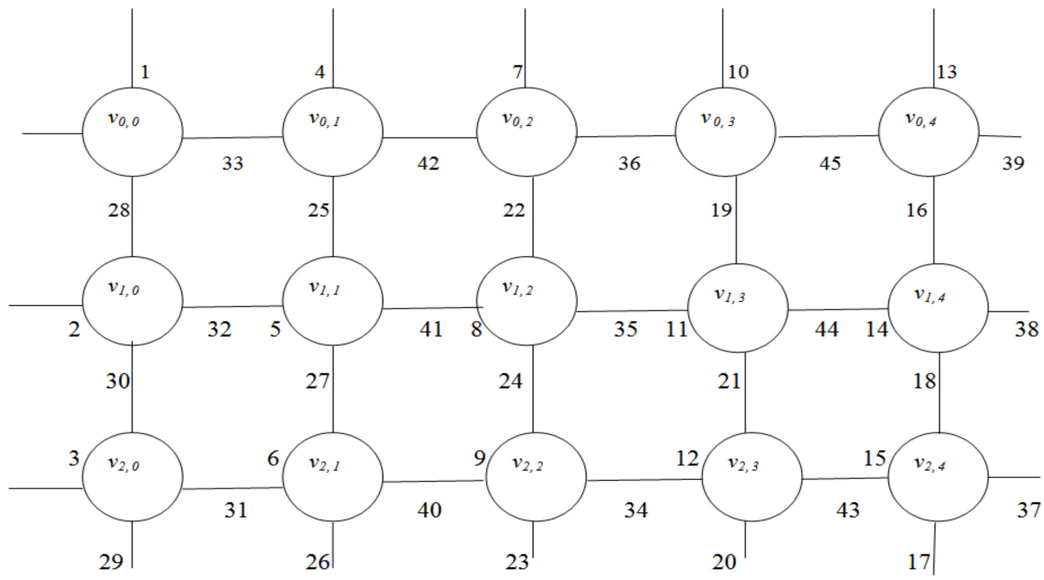

Figure 3.1: *V*-Super vertex Magic Total Labeling of $C_3 \times C_5$ with Magic Constant *h=130*

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